Chapter 8

POINTS

A point is the simplest of the elementary geometric objects: points, lines, and planes. In fact we cannot define a point in terms of anything simpler except as a set of numbers. Points are the basic building blocks for all other geometric objects, and elementary geometry demonstrates how many figures are defined as a locus of points with certain constraining characteristics. For example, in a plane, a circle is the locus of points equidistant from a given point, and a straight line is the locus of points equidistant from two given points. In three-dimensional space, a plane is the locus of points equidistant from two given points. We can also define more complex curves, surfaces, and solids this way, by using equations to define the locus of points. This is a powerful way of describing geometric objects, because it allows us to analyze and quantify their properties and relationships. Most importantly to today's technology, points are indispensable when we create computer graphic displays and geometric models. This chapter discusses the definition of a point as a set of real numbers, point relationships, arrays of points, absolute and relative points, displaying points, pixels and point resolution, and translating and rotating points.

8.1 Definition

A point suggests the idea of place or location. We define a point by a set of one or more real numbers, its coordinates. The coordinates of a point not only locate it in a coordinate system, but also with respect to other points in the system.

A set of n real numbers define a point in n-dimensional space

$$\mathbf{p} = (x_1, x_2, \ldots, x_n) \tag{8.1}$$

where $x_1, x_2, \ldots, x_n$ are the coordinates of $\mathbf{p}$ and n is the number of dimensions of the coordinate system. A boldface, lowercase letter $\mathbf{p}$ will denote a point. This is consistent with the vector notation of Chapter 1. In fact, under certain conditions there is a one-to-one correspondence between the coordinates of a point and the components of its vector representation.

In geometric modeling and computer graphics, most work is in two- or three-dimensional space, so we will not usually see Equation 8.1 in this generalized form. Note, though, that each coordinate has an identifying number, shown as an attached subscript. This is usually not necessary if the geometry is in two or three dimensions, where there are plenty of letters with which to name the coordinates, such as x, y, and z. In fact, a global or world coordinate system in computer graphics is three-dimensional, and points in it are given by

$$\mathbf{p} = (x, y, z) \tag{8.2}$$

However, we often use a subscript to identify a specific point. For example,

$$\mathbf{p}_1 = (x_1, y_1, z_1) \tag{8.3}$$

Points are a key part of the definition of a coordinate system, and we can extend the definition of a point to define a coordinate system. In three dimensions it is the set of all points defined by the triplet of real numbers (x, y, z), where $x, y, z \in (+\infty, -\infty)$. From this we can derive all the other characteristics of the coordinate system. For example, the origin is the point with $x = 0$, $y = 0$, and $z = 0$, or $\mathbf{p} = (0, 0, 0)$. The x axis is the set of points $\mathbf{p} = (x, 0, 0)$, where $x \in (+\infty, -\infty)$. A *rectangular Cartesian coordinate system* is further restricted in that the coordinate axes must be mutually perpendicular and the same distance scale must be used on all the axes.

A point has no geometric or analytic properties other than place. It has no size, orientation, length, area, or volume. It has no inside or outside, nor any other common geometric characteristics. However, things change when we have two or more points, for then we can compute and test many interesting properties describing the relationships of these points to each other. For example, if we are given two points in space $\mathbf{p}_1$ and $\mathbf{p}_2$ (Figure 8.1), we can compute the distance d between them by using the Pythagorean theorem:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{8.4}$$

Note that the distance d is always a positive real number. We can assign a specific unit of measurement to the coordinate system: centimeters, meters, light-years, and so on. The distance is then in terms of this unit of measurement. Although our choice may be arbitrary, we must be consistent throughout an application.

The coordinates of the midpoint, $\mathbf{p}_m$, between two points are

$$\mathbf{p}_m = \left[\frac{(x_1 + x_2)}{2}, \frac{(y_1 + y_2)}{2}, \frac{(z_1 + z_2)}{2}\right] \tag{8.5}$$

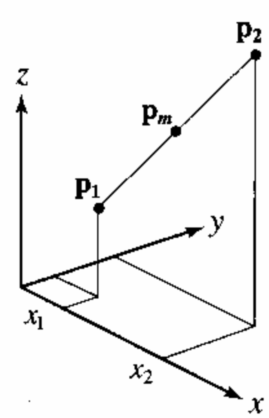

Figure 8.1 *Two points in space and their midpoint.*

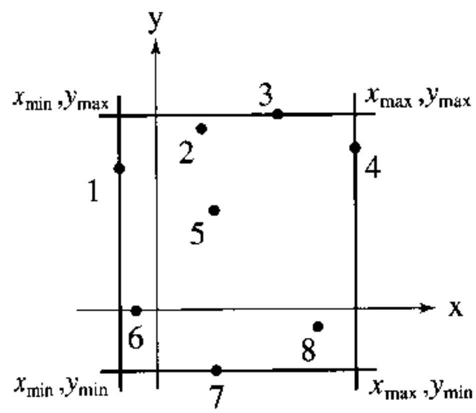

Figure 8.2 *Min-max box enclosing a set of points in the x, y plane.*

For any point in a set of points we can compute which other point is closest and which is farthest. We do this by computing and comparing d for each pair of points. If the absolute distance is not required, then d^2 will do just as well, and we avoid a relatively expensive computation.

Determining the vertices of a minimal rectangular solid, or box, which just contains all the points of a given set, is another useful computation in geometric modeling and computer graphics. This is particularly true when we must test relationships between two or more sets of points or containment of a point in a given volume of space. To compute the coordinates of the eight vertices of the box, we find the maximum and minimum x, y, and z coordinates of the point set. We assume that the edges of the box are parallel to the coordinate axes.

Figure 8.2 is an example of a *min-max* box enclosing a set of points in the x, y plane. The points shown have the following coordinates:

Point 1: $(-2, 7)$. This point has the minimum x value, written as x_{min}, in the set.

Point 2: $(2, 9)$

Point 3: $(6, 10)$. This point has the maximum y value, y_{max}.

Point 4: $(10, 8)$. This point has the maximum x value, x_{max}.

Point 5: $(3, 5)$

Point 6: $(-1, 0)$

Point 7: $(3, -3)$. This point has the minimum y value, y_{min}.

Point 8: $(8, -1)$

8.2 Arrays of Points

When a point is entered into a computer program, its coordinates are stored in an *array*. An array is an ordered arrangement of numbers which also may include an identifying number for each point. Otherwise the position of the coordinates of a point within the array may be all the "identification" needed. Figure 8.3 shows the coordinates of six points arranged in a rectangular array of 18 numbers. This is an efficient

	Column 1	Column 2	Column 3		
	x	y	z		
Row 1	8.1	–2.7	4.0	←	Point 1
Row 2	6.9	3.0	–6.5	←	Point 2
Row 3	5.7	1.1	2.1	←	Point 3
Row 4	0.0	4.2	–7.5	←	Point 4
Row 5	11.2	–0.8	–3.1	←	Point 5
Row 6	–6.6	1.0	–2.9	←	Point 6

Figure 8.3 *Coordinates of six points arranged in an array.*

way to represent and manipulate the coordinates. However, the numbers are not stored in the computer in "rectangular" configurations, as we shall see. We can present this information in a more compact form as

$$A = \begin{bmatrix} 8.1 & -2.7 & 4.0 \\ 6.9 & 3.0 & -6.5 \\ 5.7 & 1.1 & 2.1 \\ 0.0 & 4.2 & -7.5 \\ 11.2 & -0.8 & -3.1 \\ -6.6 & 1.0 & -2.9 \end{bmatrix}$$

We can identify an array with a symbol, say A, and use double subscripts to identify a particular coordinate in the array. For example, if we use A_{ij}, then the subscript i denotes the row and j denotes the column in which the number A_{ij} appears. So, in the array above, the value of A_{43} is -7.5.

Instead of being stored as a rectangular array, coordinates are usually arranged in a linear sequence, or list, of numbers. There are two ways to form a list of coordinate values for n points (Figure 8.4). Point-identifying information is often included in these data arrays, although none is shown here. In Figure 8.4a, the coordinates of each point are grouped together in sequence. In Figure 8.4b, all the x coordinate values are listed first, followed by the y and then z coordinates.

8.3 Absolute and Relative Points

In geometric modeling, or in creating a computer-graphics display, we use two kinds of points: *absolute points* and *relative points*. We make this distinction because of the way we compute the coordinates and the way we plot and display them. We define absolute points directly by their individual coordinates. For example, a set of absolute

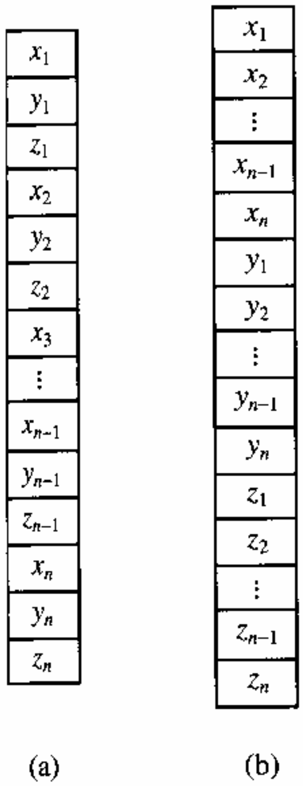

Figure 8.4 *Two ways to list point coordinates.*

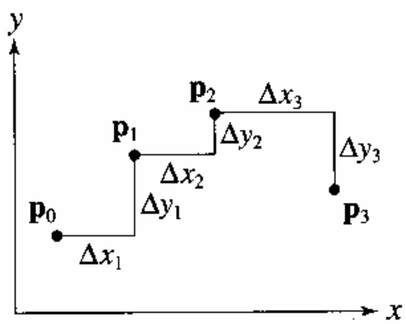

Figure 8.5 *Relative points.*

points is

$$\text{Absolute point } \mathbf{p}_i = (x_i, y_i) \quad \text{for } i = 1, \ldots, n \tag{8.6}$$

We define the coordinates of each relative point with reference to the coordinates of the point preceding it. This is best demonstrated by the following expression:

$$\text{Relative point } \mathbf{p}_i = \mathbf{p}_{i-1} + \Delta_i \quad \text{for } 1, \ldots, n \tag{8.7}$$

where $\mathbf{p}_0$ is some initial point and where $\Delta_i = (\Delta x_i, \Delta y_i)$ (Figure 8.5). This sequence may be generated during numerical analysis or when computing points on lines, curves, or surfaces.

8.4 Displaying Points

Now we must digress and discuss another geometry problem of contemporary application, again in computer graphics. A common point-display problem arises when we must define a *window* that just encloses a set of points. To do this, we investigate each point $\mathbf{p}_i$ in a search for the maximum and minimum x and y values, where $W_R = \max x$, $W_L = \min x$, $W_T = \max y$, and $W_B = \min y$, which define the window boundaries.

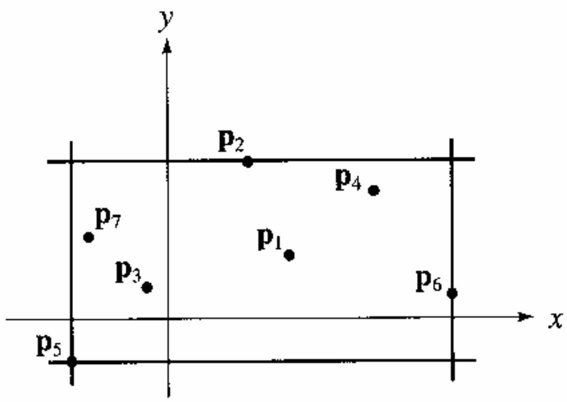

Figure 8.6 *Window boundaries and the min-max box.*

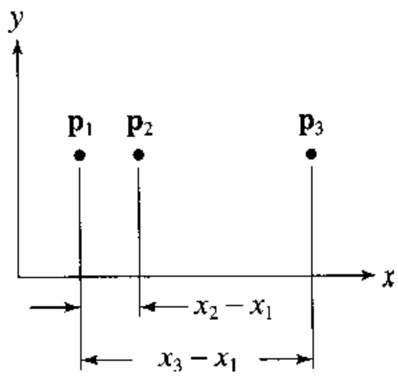

Figure 8.7 *Limit of resolution of displayed points.*

In Figure 8.6 there are seven points with coordinates $\mathbf{p}_1 = (3.0, 1.5)$, $\mathbf{p}_2 = (2.0, 4.0)$, $\mathbf{p}_2 = (-0.5, 0.7)$, $\mathbf{p}_4 = (5.0, 3.0)$, $\mathbf{p}_5 = (-2.5, -1.0)$, $\mathbf{p}_7 = (-2.0, 2.0)$. It is easy to see that $\max x = x_6$, $\min x = x_5$, $\max y = y_2$, and $\min y = y_5$. This means that $W_R = 7.0$, $W_L = -2.5$, $W_T = 4.0$, and $W_B = -1.0$. We can confirm this visually by inspecting the points in the figure.

An interesting problem may arise when displaying three or more points. It is not always possible to resolve every point in a set of points to be displayed. Let's consider, for example, the three points, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, in Figure 8.7. For simplicity, we can arrange them on a common horizontal line so that $y_1 = y_2 = y_3$. This means that only the x values determine their separation. If the number of pixels H between $\mathbf{p}_1$ and $\mathbf{p}_3$ is less than the ratio of the separations of $\mathbf{p}_1$ and $\mathbf{p}_3$ to $\mathbf{p}_1$ and $\mathbf{p}_2$, then $\mathbf{p}_1$ and $\mathbf{p}_2$ will not be resolved. That is, $\mathbf{p}_1$ and $\mathbf{p}_2$ will not be displayed as two separate and distinct points. This relationship is expressed by the inequality $(x_3 - x_1)/(x_2 - x_1) > H_{\text{pixels}}$. If this inequality is true, then $\mathbf{p}_1$ and $\mathbf{p}_2$ cannot be resolved, and they will be assigned the same pixel.

It is also important to know if a given point in the picture plane is inside or outside the window region of a computer-graphics display. If the coordinates of a point are x, y, then the point is inside the window if and only if both of the following inequalities are true:

$$W_L \le x \le W_R \quad \text{and} \quad W_B \le y \le W_T \tag{8.8}$$

8.5 Translating and Rotating Points

We can move a point from one location to another in two ways: We can *translate* it from its current position to a new one, or we can *rotate* it about some point in the plane or axis in space to a new position. A translation is described by making changes relative to a point's coordinates (Figure 8.8). These changes are denoted as x_T and y_T, and the

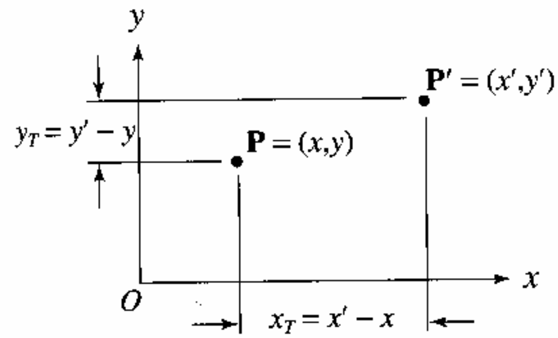

Figure 8.8 *Translating a point.*

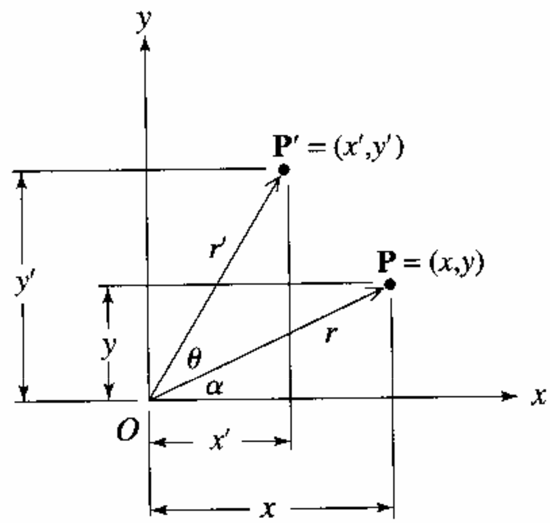

Figure 8.9 *Rotating a point about the origin.*

translation equations are

$$\begin{aligned} x' &= x + x_T \\ y' &= y + y_T \end{aligned} \tag{8.9}$$

where x' and y' are the coordinates of the new location. Generalizing these equations to three or more dimensions is straightforward. Now is a good time to review Chapter 3 on transformations.

The simplest rotation of a point in the coordinate plane is about the origin (Figure 8.9). If we rotate a point $\mathbf{p}$ about the origin and through an angle θ, then we derive the coordinates of the transformed point $\mathbf{p}'$ as follows: we express x' and y' in terms of $\alpha + \theta$ and r', thus

$$\begin{aligned} x' &= r' \cos(\alpha + \theta) \\ y' &= r' \sin(\alpha + \theta) \end{aligned} \tag{8.10}$$

From elementary trigonometry we have

$$\begin{aligned} \cos(\alpha + \theta) &= \cos\alpha\cos\theta - \sin\alpha\sin\theta \\ \sin(\alpha + \theta) &= \sin\alpha\cos\theta - \cos\alpha\sin\theta \end{aligned} \tag{8.11}$$

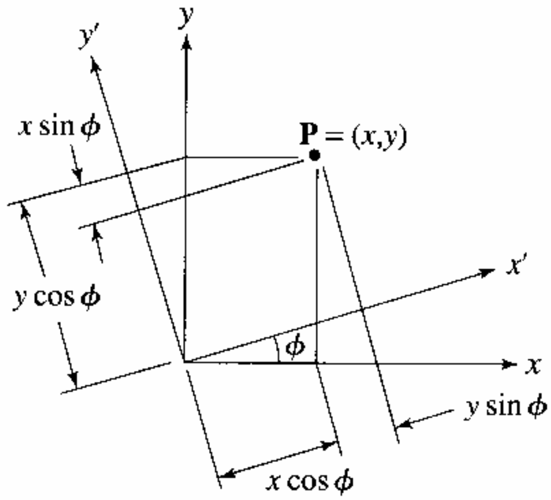

Figure 8.10 *Rotating the coordinate axes.*

and

$$\cos\alpha = \frac{x}{r} \qquad \sin\alpha = \frac{y}{r} \tag{8.12}$$

Because $r = r'$, with appropriate substitutions of the preceding equations into Equation 8.10, we find

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned} \tag{8.13}$$

This set of equations describes a rotation transformation in the plane about the origin. We can generalize this description to three dimensions if we change from rotation about the origin—a point—to rotation about a straight line, an axis of rotation. In fact, Equation 8.13 describes the rotation of a point about the z axis. We define a positive rotation in the plane as counterclockwise about the origin.

We can just as easily find the coordinates of a point in a new coordinate system which shares the same origin as the original system but which is rotated through some angle, say ϕ, with respect to it. Figure 8.10 illustrates the geometry of this transformation. The coordinates of a point in the new system are

$$\begin{aligned} x' &= x\cos\phi + y\sin\phi \\ y' &= -x\sin\phi + y\cos\phi \end{aligned} \tag{8.14}$$

Exercises

8.1 Compute the distance between each of the following pairs of points:

a. $(-2.7, 6.5, 0.8)$ and $(5.1, -5.7, 1.9)$

b. $(1, 1, 0)$ and $(4, 6, -3)$

c. $(7, -4, 2)$ and $(0, 2.7, -0.3)$

d. $(-3, 0, 0)$ and $(7, 0, 0)$

e. $(10, 9, -1)$ and $(3, 8, 3)$

8.2 Compute the coordinates of the points in Exercise 8.1 relative to a coordinate system centered at (3, 1, 0) in the original system and parallel to it.

8.3 Compute the distance between each of the points found for Exercise 8.2.

8.4 Show that the distance between any pair of points is independent of the coordinate system chosen.

8.5 Compute the midpoint between the pairs of points in Exercise 8.1.

8.6 Given an arbitrary set of points, find the coordinates of the vertices of a rectangular box that just encloses it.

8.7 Find the coordinates of the eight vertices of a rectangular box that just encloses the ten points given in Exercise 8.1.

8.8 Given that Δ_i is a constant for all $\mathbf{p}_i$, that is, $\Delta_i = (\Delta x_i, \Delta y_i) = (\Delta x, \Delta y)$, find $\mathbf{p}_4$ in terms of $\mathbf{p}_0$ and Δ_i.

8.9 Find the set of Δ_is for the vertex points of a square whose sides are three units long and with $\mathbf{p}_0 = (1, 0)$. Assume that the sides of the square are parallel to the x, y coordinate axes, and proceed counterclockwise.

8.10 Repeat Exercise 8.9 for a square whose sides are four units long and with $\mathbf{p}_0 = (-2, -2)$.

8.11 Repeat Exercise 8.10 for $\mathbf{p}_0 = (1, -4)$.

8.12 Given $W_R = 14$, $W_L = -2$, $W_T = 8$, and $W_B = -4$, determine which of the following points are inside this computer-graphics display window:

a. $\mathbf{p}_1 = (15, 4)$
b. $\mathbf{p}_2 = (3, 10)$
c. $\mathbf{p}_3 = (14, 2)$
d. $\mathbf{p}_4 = (-2, 8)$
e. $\mathbf{p}_5 = (10, 10)$

8.13 Find the coordinates of the corners of the window defined in Exercise 8.12.

8.14 State a mathematical test to determine if a point is contained in a rectangular volume in space, whose sides are parallel to the principal planes.

8.15 Derive the equations that describe the resultant transformation of a point that is first translated by x_T, y_T and then rotated by θ.

8.16 Derive the equations that describe the resultant transformations of a point that is first rotated by θ and then translated by x_T, y_T.

8.17 Compare and comment on the results of Exercises 8.15 and 8.16.